Math 210B Lecture 6 Notes

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1 Transcendental Extensions and Separability

1.1 Transcendental extensions

Definition 1.1. An extension K/F is **purely transcendental** if every $\alpha \in K \setminus F$ is transcendental over F.

Proposition 1.1. *$F((t_i)_{i \in I})$, where I is an indexing set, is purely transcendental over F.*

Proof. Here is the case of $F(t)/F$. Let $\alpha = f/g \in F(t) = F$, where $f, g \in F[t]$, and $g \neq 0$. Then $\alpha g(x) \notin F[x]$, but $\alpha g(x) \in F(t)[x]$. Then $\alpha g(x) \neq f(x) \in F[x]$. But $f(xx) - \alpha g(x)$ has a root t, so t is algebraic over $F(\alpha)$. But t is transcendental over F, so α must be transcendental over F. Thus, $F(t)/F$ is purely transcendental.

For the case of $F(t_1, \dots, t_n)/F$, proceed by induction. For the general case, every element in $F((t_i)_{i \in I})$ is in $F(t_1, \dots, t_n)$ for some $i_1, \dots, i_n \in I$. If it is not in F, it is transcendental by the previous case. $\square$

Proposition 1.2. *Every field extension is a purely transcendental extension of an algebraic extension.*

Proof. Let K/F, and let E be the maximal algebraic extension of F in K. If $\alpha \in K$ is algebraic over E, it is algebraic over F, so $\alpha \in E$. So K/E is purely transcendental. $\square$

Example 1.1. Let F be a field, and let $\overline{F}$ be an algebraic closure. Then $\overline{F}(t)/\overline{F}$ is purely transcendental. We can do it the other way around, as well. $\overline{F}(t)/F(t)$ is algebraic, while $F(t)/F$ is purely transcendental.

Definition 1.2. A subset $S \subseteq K$ for K/F is **algebraically independent** over F if for all nonzero $f \in F[x_1, \dots, x_n]$ and distinct $s_1, \dots, s_n \in S$, $f(s_1, \dots, s_n) \neq 0$.

Here are some lemmas about algebraically independent sets. The proofs are the same as the corresponding properties of linearly independent sets.

Lemma 1.1. *Let $S \subseteq K$ be algebraically independent over F. Then $t \in K$ is transcendental over $F(S)$, where $F(S)$ is the smallest subfield of K generated by S over F, if and only if $S \cup \{t\}$ is algebraically independent over F.*

Lemma 1.2. *$S \subseteq K$ is algebraically independent over F if and only if every $s \in S$ is transcendental over $F(S \setminus \{s\})$.*

Definition 1.3. A subset S of K is a **transcendence basis** for K/F if it is algebraically independent over F and if $K/F(S)$ is algebraic.

Example 1.2. Let $\overline{F}(t)/F$. $\{r\}$ is a transcendence basis, and in fact, $\{t^{1/n}\}$ is a trascendence basis for any n. However $\{t^{1/2}, t^{1/3}\}$ is not because it is not algebraically independent: $(t^{1/2})^2 = (t^{1/3})^3$.

The previous two lemmas imply the following lemma.

Lemma 1.3. *Let $S \subseteq K$. The following are equivalent:*

1. *S is a trascnece basis for K/F.*
2. *S is a maximal F-algebraically independent subset of K.*
3. *S is a minimal subset of K such that K is algebraic over $F(S)$.*

Proof. The first two statements are equivalent by the first lemma. The latter two statements are equivalent by the second. □

Theorem 1.1. *Every F-algebraiclly independent subset of K is contained in a transcendence basis, and every $S \subseteq K$ such that $K/F(s)$ is algebraic contains a trascendence basis.*

The proof is the same argument as the corresponding statement in linear algebra.

Corollary 1.1. *Every field extension has a transcendence basis. In particular, there exists an intermediate extension $K/E/F$ such that K/E is algebraic and E/F is purely transencental.*

Proof. Take $E = F(S)$, where S is a transcendence basis. □

Theorem 1.2. *Any two transcendence bases of K/F have the same cardinality.*

Again, the proof is the same as the corresponding proof in linear algebra.

Definition 1.4. The **transcendence degree** of K/F is the number of elements in a transcendence bases if finite. Otherwise, K/F has infinite transcendence degree.

1.2 Separability

Definition 1.5. Let $f \in F[x]$. The **multiplicity** of a root α of F in an algebraic closure of F is the highest power m such that $(x-\alpha)^m \mid f$ in $\overline{F}[x]$.

Example 1.3. The polynomial $x^p - t = (x - t^{1/p})^p$ in $\mathbb{F}_p(t^{1/p})[x]$. The multiplicity of $t^{1/p}$ is p.

Lemma 1.4. *The multiplicity of a root odes not depend on the choice of $\overline{F}$ and does not depend on the choice of root if f is irreducible.*

Corollary 1.2. *The number of distinct roots in $\overline{F}$ of an irredudcible polynomial $f \in F[x]$ divides* $\deg(f)$.

Proof. Write $f = \prod_{i=1}^{k}(x-\alpha_i)^m$. Then $km = \deg(f)$. $\square$

Definition 1.6. We say that $f \in F[x]$ is **separable** if every root of f has multiplicity 1. An element $\alpha \in \overline{F}$ is **separable** if it is algebraic over F and its minimal polynomial over F is separable. An extension E/F is **separable** if every $\alpha \in E$ is separable over F.

Lemma 1.5. *Let E/F be a field extension and $\alpha \in E$ be algebraic over F. Then α is separable over F if and only if $F(\alpha)/F$.*

Proof. If $F(\alpha)/F$ is separable, then $\alpha \in F(\alpha)$, so α is separable over F. Conversely, suppose α is separable over F, and let $\beta \in F(\alpha)$. The number of embeddings of $F9\beta \int \overline{F}$ fixing F is $\leq [F(\beta) : F]$. Equality holds iff β is separable over F.

The number of embeddings $F(\alpha) \to \overline{F}$ is $[F(\alpha) : F]$. On the other hand, α is separable over $F(\beta)$, so the number of embeddings $F(\alpha) \to \overline{F}$ extending the embedding $F(\beta) \to \overline{F}$ equals $[F(\alpha) : F(\beta)]$. So the number of embeddings $F(\alpha) \to \overline{F}$ over F is the product of the number of embeddings $F(\beta) \to \overline{F}$ with the number of extensions of these embeddings to $F(\alpha) \to \overline{F}$. So the number of embeddings $F(\beta) \to \overline{F}$ fixing F is

$$\frac{[F(\alpha) : F]}{[F(\alpha) : F(\beta)]} = [F(\beta) : F]. \qquad \square$$